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The natural mappings i_n and k-subspaces of free topological groups on metrizable spaces

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Dedicated to Professor Takao Hoshina on his sixtieth birthday

Abstract

Let $F(X)$ be the free topological group on a Tychonoff space X. For all natural number n we denote by $F_n(X)$ the subset of $F(X)$ consisting of all words of reduced length $\leqslant n$, and by i_n the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ to $F_n(X)$. We prove that for a metrizable space X if $F_n(X)$ is a k-space for each n, then X is locally compact and either separable or discrete. Therefore, as a corollary, we obtain that for a metrizable space X if $F_n(X)$ is a k-space for all $n \in \mathbb{N}$, then so is $F(X)$. Furthermore, it is proved that for a metrizable space X the following are equivalent: (i) the mapping i_n is a quotient mapping for each n; (ii) a subset U of $F(X)$ is open if $i_n^{-1}(U \cap F_n(X))$ is open in $(X \oplus X^{-1} \oplus \{e\})^n$ for each n; (iii) X is locally compact separable or discrete.

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1. Introduction

Let $F(X)$ and $A(X)$ be respectively the *free topological group* and the *free Abelian topological group* on a Tychonoff space X in the sense of Markov [7]. As an abstract group, $F(X)$ is free on X and it carries the finest group topology that induces the original topology of X, in other words, every continuous map from X to an arbitrary topological group lifts in a unique fashion to a continuous homomorphism from $F(X)$. Similarly, as an abstract group, $A(X)$ is the free Abelian group on X, having the finest group topology that

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induces the original topology of X, so that every continuous map from X to an arbitrary Abelian topological group extends to a unique continuous homomorphism from $A(X)$.

For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose length is less than or equal to n. It is known that X itself and each $F_n(X)$ are closed in $F(X)$. The subspace $A_n(X)$ is defined similarly and each $A_n(X)$ is closed in $A(X)$. Let e be the identity of $F(X)$ and 0 be that of $A(X)$. For each $n \in \mathbb{N}$ and an element $(x_1, x_2, \dots, x_n)$ of $(X \oplus X^{-1} \oplus \{e\})^n$ we call $x_1 x_2 \cdots x_n$ a *form*. In the Abelian case, $x_1 + x_2 + \cdots + x_n$ is also called a *form* for $(x_1, x_2, \dots, x_n) \in (X \oplus -X \oplus \{0\})^n$. We remark that a form may contain some reduced letter. Then the reduced form of $x_1 x_2 \cdots x_n$ is a word of $F(X)$ and that of $x_1 + x_2 + \cdots + x_n$ is a word of $A(X)$. For each $n \in \mathbb{N}$ we denote the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ onto $F_n(X)$ by i_n and we also use the same symbol i_n in the Abelian case, that is, i_n means the natural mapping from $(X \oplus -X \oplus \{0\})^n$ onto $A_n(X)$. Clearly the mapping i_n is continuous for each $n \in \mathbb{N}$.

The following problems have been studied by several mathematicians and described in [10].

Problem 1. Characterize spaces X for which the mapping i_n is a quotient (closed, z-closed, R-quotient, etc.) mapping for all $n \in \mathbb{N}$.

Problem 2. Find general conditions on X implying that $F(X)$ (or $F_n(X)$ for each $n \in \mathbb{N}$) is a k-space.

Problem 1 was completely solved for $n = 2$ by Pestov [8]. He proved that i_2 is a quotient mapping iff X is strongly collectionwise normal, i.e., if every neighborhood of the diagonal in X^2 contains a uniform neighborhood of the diagonal. Furthermore, the author [13] proved that i_2 is a quotient mapping iff i_2 is closed. The author also proved in the same paper that for a metrizable space X the mapping i_n is closed for each $n \in \mathbb{N}$ iff X is compact or discrete. They are true for Abelian case.

The author [12] obtained a characterization of a metrizable space such that every i_n is a quotient mapping for Abelian case. He proved that for a metrizable space X, i_n for Abelian case is a quotient mapping for each $n \in \mathbb{N}$ if and only if either X is locally compact and the set dX of all nonisolated points in X is separable, or dX is compact. As the author mentioned in [12, Proposition 4.1], for a Dieudonné complete (and hence, metrizable) space X, i_n is a quotient mapping iff $A_n(X)$ ($F_n(X)$) is a k-space for each $n \in \mathbb{N}$. So, the above result is also an answer to Problem 2 for the free Abelian topological group on a metrizable space.

The aim of this paper is to solve the above problems for the *non-Abelian* free topological group on a metrizable space. As a consequence, we can know whether each $i_n : (X \oplus X^{-1} \oplus \{e\})^n \to F_n(X)$ is a quotient mapping or not, and hence whether each $F_n(X)$ is a k-space or not for such familiar metric spaces X as the real line $\mathbb{R}$, the space $\mathbb{Q}$ of rational numbers, $\mathbb{R} \setminus \mathbb{Q}$, $J(\kappa)$ ($\kappa \geqslant \omega$) be the hedgehog space of spine κ such that each spine is a sequence which converges to the center point or the topological sum C_κ of $\kappa\,(\geqslant \omega)$ many convergent sequences with their limits.

We first show that for a metrizable space X if i_n for non-Abelian case is a quotient mapping for each $n \in \mathbb{N}$, then X is locally compact separable or discrete. Then we shall give a characterization of a metrizable space X such that each $F_n(X)$ is a k-space, and hence each i_n is a quotient mapping.

On the other hand, the situation is different for $n = 3$. In [12], the author proved that for a metrizable space X the following are equivalent: (1) $A_3(X)$ is a k-space; (2) i_3 is a quotient mapping; (3) X is locally compact or dX is compact. In this paper, we add the conditions (1′) $F_3(X)$ is a k-space and (2′) i_3 is a quotient mapping for non-Abelian case to the lists of equivalences in the above result.

In the last section, we consider a simple description of the topology of $F(X)$ $(A(X))$, as follows;

$$\begin{aligned}&\text{a set } U \subseteq F(X) \text{ is open in } F(X) \quad \text{if and only if}\\ &i_n^{-1}\big(U \cap F_n(X)\big) \text{ is open in } \big(X \oplus X^{-1} \oplus \{e\}\big)^n \text{ for each } n \in \mathbb{N},\\ &\text{a set } U \subseteq A(X) \text{ is open in } A(X) \quad \text{if and only if}\\ &i_n^{-1}\big(U \cap A_n(X)\big) \text{ is open in } \big(X \oplus -X \oplus \{0\}\big)^n \text{ for each } n \in \mathbb{N}.\end{aligned}$$

Clearly, if $F(X)$ has the inductive limit topology and i_n is a quotient mapping for each $n \in \mathbb{N}$, then $F(X)$ has the above description. On the other hand, since the mapping i_n is continuous, if $F(X)$ has the above description, then $F(X)$ has the inductive limit topology. In this paper, we shall show that if $F(X)$ has the above description, then i_n is a quotient mapping for each $n \in \mathbb{N}$. As a consequence, we obtain that for a Dieudonné complete space X, $F(X)$ is a k-space iff $F(X)$ has the above simple description. Furthermore, from the above theorem, we can prove that for a metrizable space X, $F(X)$ has the above simple description iff X is locally compact separable or discrete. Since the above argument is also true for Abelian case, we obtain that for a Dieudonné complete space X $A(X)$ is a k-space iff $A(X)$ has the above simple description (for $A(X)$), and for a metrizable space X, $A(X)$ has the above simple description (for $A(X)$) iff X is locally compact and the set of all nonisolated points of X is separable.

All topological spaces are assumed to be Tychonoff. By $\mathbb{N}$ we denote the set of all positive natural numbers. Our terminology and notations follow [3]. We refer to [6] for elementary properties of topological groups and to [1,4] for the advanced properties of free topological groups.

2. Characterizations

In this section, we shall show that for a metrizable space X if $F(X)$ is a k-space, then X is locally compact separable or discrete. To prove it, we need the description of a neighborhood base of e in $F(X)$ obtained by Uspenskiĭ [11]. Let $P(X)$ be the set of all continuous pseudometrics on a space X. Put

$$F_0(X) = \left\{h = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{2n}^{\varepsilon_{2n}} \in F(X)\colon \sum_{i=1}^{2n} \varepsilon_i = 0,\ x_i \in X \text{ for } i = 1, 2, \ldots, n,\ n \in \mathbb{N}\right\}$$

Then $F_0(X)$ is a clopen normal subgroup of $F(X)$. It is well known that every $h \in F_0(X)$ can be represented as

$$h = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} g_2 x_2^{\varepsilon_2} y_2^{-\varepsilon_2} g_2^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_n} g_n^{-1}$$

for some $n \in \mathbb{N}$, where $x_i, y_i \in X$, $\varepsilon_i = \pm 1$ and $g_i \in F(X)$ for $i = 1, 2, \ldots, n$. Take an arbitrary $r = \{\rho_g\colon g \in F(X)\} \in P(X)^{F(X)}$. Let

$$p_r(h) = \inf\left\{\sum_{i=1}^{n} \rho_{g_i}(x_i, y_i)\colon h = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_n} g_n^{-1},\ n \in \mathbb{N}\right\}$$

for each $h \in F_0(X)$. Then Uspenskiĭ [11] proved that

(1) p_r is a continuous seminorm on $F_0(X)$ and
(2) $\{\{h \in F_0(X)\colon p_r(h) < \delta\}\colon r \in P(X)^{F(X)}, \delta > 0\}$ is a neighborhood base of e in $F(X)$. (Note that $p_r(e) = 0$ for each $r \in P(X)^{F(X)}$.)

Therefore, to prove $e \in \overline{E}$ for some $E \subseteq F(X)$, it suffices to show that for each $r \in P(X)^{F(X)}$ there is $h \in E \cap F_0(X)$ such that $p_r(h) < 1$.

For each word $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \in F(X)$ let $\operatorname{car}(g) = \{x_1, x_2, \ldots, x_n\}$, and let $\operatorname{car} A = \bigcup\{\operatorname{car}(g)\colon g \in A\}$ for a subset A of $F(X)$. Arhangel'skiĭ et al. [2] proved that if E is a bounded (in particular, compact) set in $F(X)$, then $\operatorname{car} E$ is bounded in X.

Propposition 2.1. *For a metrizable space X if $F_n(X)$ is a k-space for each $n \in \mathbb{N}$, then X is locally compact.*

Proof. Suppose that X is not locally compact. Then X contains the hedgehog space J of spine ω such that each spine is a sequence which converges to the center point as a closed subset. So, it suffices to prove that $F_n(J)$ is not a k-space for some $n \in \mathbb{N}$. Indeed, we shall show that $F_5(J)$ is not a k-space.

Denote the hedgehog space J by $J = \{x\} \cup \bigcup_{n \in \omega} X_n \oplus \{z_i\colon i \in \omega\}$, where

(1) $X_n = \{y_n\} \cup \{x_{n,j}\colon j \in \omega\}$ is a closed discrete subset of J for each $n \in \omega$,
(2) $\{z_i\colon i \in \omega\}$ is a closed discrete subset of J, and
(3) $\{\{x\} \cup \bigcup_{n \geqslant k} X_n\colon k \in \omega\}$ is a neighborhood base of x in J.

Put $E = \{g_{n,j} = z_n^{-1} y_j^{-1} x z_n x_{n,j}\colon n, j \in \omega\} \subseteq F_5(J)$. If we show that $E \cap K$ is closed in $F_5(J)$ for each compact set $K \subseteq F_5(J)$ and $x \in \overline{E} \setminus E$, then we can conclude that $F_5(J)$ is not a k-space.

Let K be a compact subset of $F_5(J)$. Then $\operatorname{car} K$ is bounded in J. It follows, from the definition of the set E, that $\operatorname{car}(E \cap K)$ is a finite set. Then $E \cap K$ is finite and hence, it is closed in $F_5(J)$.

To show that $x \in \overline{E}$, we shall show that $e \in \overline{\{x^{-1} g_{n,j}\colon n, j \in \omega\}}$ applying Uspenskiĭ's neighborhood base of e. Take an arbitrary $r = \{\rho_g\colon g \in F(X)\} \in P(X)^{F(X)}$. Since ρ_e is a continuous pseudometric on J and by the form of the neighborhood base of x in J, there is $n_0 \in \omega$ such that $\rho_e(x, x_{n,j}) < \frac{1}{2}$ for each $n \geqslant n_0$ and $j \in \omega$. Furthermore, since $\rho_{z_{n_0} x}$ is a continuous pseudometric on J and the sequence $\{y_n\}$ converges to x, there is $j_0 \in \omega$ such that $\rho_{z_{n_0}x}(y_j, x) < \frac{1}{2}$ for each $j \geqslant j_0$. Hence we have that

$$\begin{aligned} p_r\big(x^{-1}g_{n_0,j_0}\big) &= p_r\big(x^{-1}z_{n_0}^{-1}y_{j_0}^{-1}xz_{n_0}x_{n_0,j_0}\big) \\ &= p_r\big(x^{-1}z_{n_0}^{-1}y_{j_0}^{-1}xz_{n_0}xx^{-1}x_{n_0,j_0}\big) \\ &\leqslant \rho_{z_{n_0}x}(y_{j_0}, x) + \rho_e(x, x_{n_0,j_0}) \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

This means that $x \in \overline{E}$. Since $x \notin E$, we can conclude that $F_5(X)$ is not a k-space. □

Gruenhage [5, Lemma 5] proved that the square of the sequential fan S_{ω_1} is not a k-space. To prove the following Proposition, we apply his idea used in the proof.

Propposition 2.2. *For a metrizable space X if $F_n(X)$ is a k-space for each $n \in \mathbb{N}$, then X is separable or discrete.*

Proof. Suppose that X is neither separable nor discrete. Then X contains a space $T = C \oplus D$ as a closed subset, where $C = \{x_n\colon n \in \omega\} \cup \{x\}$ is a convergent sequence with its limit x and $D = \{d_\alpha\colon \alpha \in \omega_1\}$ is a discrete closed subset of X.

For each $\alpha \in \omega_1$ let $f_\alpha : \omega_1 \to \omega$ be a function such that $f_\alpha|_\alpha : \alpha \to \omega$ is a bijection. Let

$$H_{\alpha,\beta} = \big\{d_\beta x_m x^{-1} d_\beta^{-1} d_\alpha x_{f_\alpha(\beta)} x^{-1} d_\alpha^{-1}\colon m \leqslant f_\alpha(\beta)\big\}$$

for each $\alpha, \beta \in \omega_1$ and $H = \bigcup_{\alpha,\beta\in\omega_1} H_{\alpha,\beta}$. Then each $H_{\alpha,\beta}$ is a finite set and H is a subset of $F_8(T)$. To complete our proof, it suffices to show that $H \cap K$ is closed in $F_8(T)$ for each compact subset K of $F_8(T)$ and H is not closed in $F_8(T)$.

Let K be a compact subset of $F_8(T)$. Since $\operatorname{car} K$ is bounded in T, $(\operatorname{car} K) \cap D$ is a finite set. That is, there is a finite set $F \subseteq \omega_1$ such that $H \cap K \subseteq \bigcup_{\alpha,\beta\in F} H_{\alpha,\beta}$. Hence, $H \cap K$ is closed in $F_8(T)$.

To show that H is not closed in $F_8(T)$, we apply Uspenskiĭ's neighborhood base of e again and show that $e \in \overline{H} \setminus H$. Let $r = \{\rho_g\colon g \in F(T)\} \in P(T)^{F(T)}$. Since the sequence $\{x_n\}$ converges to x and ρ_{d_α} is a continuous pseudometric on T for each $\alpha \in \omega_1$, there is $n(\alpha) \in \omega$ such that $\rho_{d_\alpha}(x, x_n) < \frac{1}{2}$ for each $n \geqslant n(\alpha)$. Hence, we can find $n_0 \in \omega$ and an uncountable set $A \subseteq \omega_1$ such that $\rho_{d_\alpha}(x, x_n) < \frac{1}{2}$ whenever $n \geqslant n_0$ and $\alpha \in A$. Pick $\alpha \in A$ which has infinitely many predecessors in A. Since $f_\alpha(\alpha \cap A)$ is an infinite set, there are $m > n$ and $\beta \in \alpha \cap A$ such that $f_\alpha(\beta) = m$. Hence the word $g = d_\beta x_m x^{-1} d_\beta^{-1} d_\alpha x_{f_\alpha(\beta)} x^{-1} d_\alpha^{-1}$ is in $H_{\alpha,\beta}$. On the other hand

$$\begin{aligned} p_r(g) &= p_r\big(d_\beta x_m x^{-1} d_\beta^{-1} d_\alpha x_{f_\alpha(\beta)} x^{-1} d_\alpha^{-1}\big) \\ &\leqslant \rho_{d_\beta}(x_m, x) + \rho_{d_\alpha}(x_m, x) \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Therefore, it follows that $e \in \overline{H}$. Since $e \notin H$, we can prove that H is not closed in $F_8(T)$. □

As is mentioned in Section 1, for a metrizable space X, i_n is a quotient mapping iff $F_n(X)$ is a k-space for each $n \in \mathbb{N}$. Arhangel'skiĭ et al. [2] proved that for a metrizable space X, $F(X)$ is a k-space iff X is locally compact separable or discrete. Furthermore, Pestov and the author [9] showed that for a metrizable space X, $F(X)$ is a k-space iff $F(X)$ has the inductive limit topology, i.e., a subset U of $F(X)$ is open if each $U \cap F_n(X)$ is open in $F_n(X)$. Then, from the above Propositions, we obtain an answer to the problems in Section 1 for the free topological group on a metrizable space.

Theorem 2.3. *For a metrizable space X, the following are equivalent*:

(1) *$F(X)$ is a k-space*;
(2) *$F(X)$ has the inductive limit topology*;
(3) *$F_n(X)$ is a k-space for each $n \in \mathbb{N}$*;
(4) *i_n is a quotient mapping for each $n \in \mathbb{N}$*;
(5) *X is locally compact separable or discrete.*

As compared with the Abelian case, the above result is interesting. For, it is known that for a metrizable space X,

(a) Arhangel'skiĭ et al. [2] $A(X)$ is a k-space iff X is locally compact and dX is separable, and
(b) Yamada [12] each $A_n(X)$ is a k-space iff either X is locally compact and dX is separable, or dX is compact.

Hence, there is a metrizable space X (for example, the hedgehog space such that each spin is a sequence which converges to the center point) such that each $A_n(X)$ is a k-space, and hence i_n for Abelian case is a quotient mapping, but $A(X)$ is not a k-space. On the other hand, for non-Abelian case, the above result shows that there is not such a metrizable space.

3. $F_3(X)$ and i_3

In this section, we shall give a necessary and sufficient condition of a metrizable space X in order that $F_3(X)$ is a k-space, and hence i_3 is a quotient mapping.

Let X be a space and $\overline{X} = X \oplus X^{-1} \oplus \{e\}$. For a set U of $\overline{X}^2$ which includes the diagonal of $\overline{X}^2$, put

$$W_2(U) = \{g \in F_4(X)\colon\ g \text{ can be represented as a form } x_1x_2x_3x_4 \text{ such that } (x_1, x_2^{-1}), (x_3, x_4^{-1}) \in U \text{ or } (x_1, x_4^{-1}), (x_2, x_3^{-1}) \in U\}.$$

Then the author [13] obtained the following results.

Lemma 3.1. *Let X be a space and $\mathcal{U}_{\overline{X}}$ be the universal uniformity on $\overline{X}$. Then*

(1) *$W_2(U)$ is a neighborhood of e in $F_4(X)$ for each $U \in \mathcal{U}_{\overline{X}}$, and*
(2) *$\{gW_2(U) \cap F_3(X)\colon\ U \in \mathcal{U}_{\overline{X}}\}$ is a neighborhood base of g for each $g \in X \cup X^{-1}$.*

Propposition 3.2. *For a metrizable space X if $F_3(X)$ is a k-space, then X is locally compact or the set of all nonisolated points of X is compact.*

Proof. Suppose that neither X is locally compact nor the set of all nonisolated points of X is compact. Then X contains a closed subset $M = \{x\} \cup \bigcup_{n\in\omega} X_n \oplus \bigoplus_{n\in\omega} C_n$, where for each $n \in \omega$

$$X_n = \{x_{n,i}\colon\ i \in \omega\} \text{ is a closed discrete subset of } M,$$

$$C_n = \{c_{n,i}\colon\ i \in \omega\} \cup \{c_n\} \text{ is a convergent sequence with its limit } c_n, \text{ and}$$

$$\Big\{\{x\} \cup \bigcup_{m\geqslant n} X_m\colon\ n \in \omega\Big\} \text{ is a neighborhood base of } x \text{ in } M.$$

Set $E = \{g_{n,j} = c_n c_{n,j}^{-1} x_{n,j}\colon\ n, j \in \omega\}$. Then E is a subset of $F_3(X)$. To complete the proof, we shall show that $F_3(X)$ is not a k-space, that is

$$E \cap K \text{ is closed in } F_3(X) \text{ for every compact subset } K \text{ of } F_3(X), \text{ and}$$

$$x \in \overline{E} \setminus E, \text{ and hence } E \text{ is not closed in } F_3(X).$$

With the same argument of the proofs of previous propositions in Section 2, we can show that for every compact subset K of $F_3(X)$, $E \cap K$ is a finite set, and hence closed in $F_3(X)$. So, it suffices to show that $x \in \overline{E}$.

Since M is closed in X $F_3(M)$ is closed in $F_3(X)$. We shall indeed show that $x \in \overline{E \cap F_3(M)}^{F_3(M)}$. Put $M_1 = \{x\} \cup \bigcup_{n\in\omega} X_n$ and $M_2 = \bigoplus_{n\in\omega} C_n$. Given $f \in \omega^\omega$ let

$$\begin{aligned} U_f = {} & \Big(\{x\} \cup \bigcup_{m\geqslant f(0)} X_m\Big)^2 \cup \Delta_{M_1} \cup \bigcup_{n\in\omega} \big(\{c_n\} \cup \{c_{n,i}\colon\ i \geqslant f(n)\}\big)^2 \cup \Delta_{M_2} \\ & \cup \Big(\{x^{-1}\} \cup \bigcup_{m\geqslant f(0)} X_m^{-1}\Big)^2 \cup \Delta_{M_1^{-1}} \cup \bigcup_{n\in\omega} \big(\{c_n^{-1}\} \cup \{c_{n,i}^{-1}\colon\ i \geqslant f(n)\}\big)^2 \\ & \cup \Delta_{M_2^{-1}} \cup \{(e, e)\}, \end{aligned}$$

where Δ_Y means the diagonal of Y^2 for $Y \in \{M_1, M_1^{-1}, M_2, M_2^{-1}\}$. Then $\{U_f\colon\ f \in \omega^\omega\}$ is a base of the universal uniformity of M. Hence, by Lemma 3.1(2), $\{xW_2(U_f) \cap F_3(M)\colon\ f \in \omega^\omega\}$ is a neighborhood base of x in $F_3(M)$.

Let $f \in \omega^\omega$ and $n, j \in \omega$ with $n \geqslant f(0)$ and $j \geqslant f(n)$. Then $(x^{-1}, x_{n,j}^{-1}) \in (\{x^{-1}\} \cup X_n^{-1})^2 \subseteq U_f$ and $(c_n, c_{n,j}) \in (\{c_n\} \cup \{c_{n,i}\colon\ i \geqslant f(n)\})^2 \subseteq U_f$. It follows that $x^{-1} g_{n,j} = x^{-1} c_n c_{n,j}^{-1} x_{n,j} \in W_2(U_f)$. Therefore $g_{n,j} = x x^{-1} g_{n,j} \in x W_2(U_f) \cap F_3(M)$. The argument shows that $x \in \overline{E \cap F_3(M)}^{F_3(M)}$. Consequently, we have that $F_3(X)$ is not a k-space. □

Propposition 3.3. *If a metrizable space X is locally compact or the set of all nonisolated points of X is compact, then $F_3(X)$ is a k-space.*

Proof. Let X be a metrizable space. The author proved, in [13, Theorem 4.11], that if the set of all nonisolated points of X is compact, then $F_3(X)$ is first countable. So we need to show that $F_3(X)$ is a k-space if X is locally compact.

Assume that X is locally compact. To complete the proof, let $E \subseteq F_3(X)$ such that $E \cap K$ is closed in $F_3(X)$ for each compact subset K of $F_3(X)$ and $g \in \overline{E}$. Our purpose is to show that $g \in E$.

If $g \in (F_2(X) \setminus F_1(X)) \cup \{e\}$, then $g \in \overline{E \cap ((F_2(X) \setminus F_1(X)) \cup \{e\})}^{F_2(X)}$, because $(F_2(X) \setminus F_1(X)) \cup \{e\}$ is open in $F_3(X)$ and is contained in the closed subset $F_2(X)$ of $F_3(X)$. As we mentioned in Section 1, i_2 is a quotient mapping, and hence $F_2(X)$ is a k-space. It is also known that $(F_2(X) \setminus F_1(X)) \cup \{e\}$ is closed in $F_2(X)$. Hence we can prove that $g \in E$. Furthermore, it is well known that $F_3(X) \setminus F_2(X)$ is homeomorphic to a subspace of $\overline{X}^3$. Hence $F_3(X) \setminus F_2(X)$ is metrizable, and hence it is a k-space. Then, in the same way to the above argument, we can show that $g \in E$ if $g \in F_3(X) \setminus F_2(X)$. Therefore we may assume that $g \in X \cup X^{-1}$. Since $(F_3(X) \setminus F_2(X)) \cup (F_1(X) \setminus \{e\})$ is open in $F_3(X)$, $g \in \overline{E \cap ((F_3(X) \setminus F_2(X)) \cup (F_1(X) \setminus \{e\}))}$. However $F_1(X) \setminus \{e\} = X \oplus X^{-1}$ is metrizable, we can show that $g \in E$ if $g \in \overline{E \cap (F_1(X) \setminus \{e\})}$.

Consequently, from the above argument, it is enough to show that $g \in E$ in the case of $g = x^{-1}$, where $x \in X$ and $E \subseteq F_3(X) \setminus F_1(X)$. (We can show similarly if $g \in X$.)

Since X is a locally compact metrizable space, we can represent X as $X = \bigoplus\{X_\alpha\colon \alpha \in A\}$, where each X_α is locally compact and separable. Let $\mathcal{U}_\alpha$ be the universal uniformity on X_α for each $\alpha \in A$. Then

$$\mathcal{U} = \left\{\bigoplus_{\alpha \in A} U_\alpha \oplus \bigoplus_{\alpha \in A} U_\alpha^{-1} \oplus \{(e,e)\}\colon\ U_\alpha \in \mathcal{U}_\alpha\right\}$$

is the universal uniformity on $\overline{X}$. So, by Lemma 3.1(2), $\{gW_2(U) \cap F_3(X)\colon\ U \in \mathcal{U}\}$ is a neighborhood base of g in $F_3(X)$. Put

$$V_2(U) = \{xx_2x_3x_4\colon\ (x, x_2^{-1}), (x_3, x_4) \in U \text{ or } (x, x_4^{-1}), (x_2, x_3^{-1}) \in U\},$$

where $U \in \mathcal{U}$. Then, since $gW_2(U) \cap F_3(X) = gV_2(U)$, $\{gV_2(U)\colon\ U \in \mathcal{U}\}$ is a neighborhood base of g in $F_3(X)$. Pick $U_0 \in \mathcal{U}$ and let

$$H = \{x_2x_3x_4 \in E\colon\ xx_2x_3x_4 \in V_2(U_0)\}.$$

Since $H = E \cap gV_2(U_0)$ and $gV_2(U_0)$ is a neighborhood of g in $F_3(X)$, $g \in \overline{H}$. Pick the unique $\alpha(x) \in A$ with $x \in X_{\alpha(x)}$. For each $xx_2x_3x_4 \in V_2(U_0)$, $(x, x_2^{-1}) \in U_0$ or $(x, x_4^{-1}) \in U_0$. Hence, from the construction of the uniformity $\mathcal{U}$, it follows that $x_2^{-1} \in X_{\alpha(x)}$ or $x_4^{-1} \in X_{\alpha(x)}$. So, if we put $H_1 = \{x_2x_3x_4 \in H\colon\ x_2^{-1} \in X_{\alpha(x)}\}$ and $H_2 = \{x_2x_3x_4 \in H\colon\ x_4^{-1} \in X_{\alpha(x)}\}$, then $H = H_1 \cup H_2$. Since $g \in \overline{H}$, we assume that $g \in \overline{H_1}$. (We can prove with the same way if $g \in \overline{H_2}$.) Represent the set H_1 as $H_1 = \{x_\lambda y_\lambda z_\lambda\colon\ x_\lambda \in X_{\alpha(x)},\ \lambda \in \Lambda\}$. Then we can show the following properties;

(1) $x \in \overline{\{x_\lambda\colon\ \lambda \in \Lambda\}}^{X_{\alpha(x)}}$ and
(2) for each $U \in \mathcal{U}$ there is $\lambda \in \Lambda$ such that $(y_\lambda, z_\lambda^{-1}) \in U$.

The second property means that $\overline{Q} \cap \Delta_X \neq \emptyset$, where $Q = \{(y_\lambda, z_\lambda^{-1})\colon\ \lambda \in \Lambda\}$. Let $\{B_i\colon\ i \in \omega\}$ be a countable neighborhood base of x in $X_{\alpha(x)}$ such that $B_0 = X_{\alpha(x)}$ and $B_{i+1} \subseteq B_i$ for each $i \in \omega$. For each $i \in \omega$, let $P_i = \{(x, x_\lambda^{-1})\colon\ x_\lambda \in B_i \setminus B_{i+1}\}$ and

$Q_i = \{(y_\lambda, z_\lambda^{-1})\colon\ x_\lambda \in B_i \setminus B_{i+1}\}$. Then, $Q = \bigcup_{i\in\omega} Q_i$ and the property (1) yields that $\overline{\bigcup_{i\in\omega} P_i} \cap \Delta_{X_{\alpha(x)}} \neq \emptyset$. Since $\overline{\bigcup_{i\in\omega} Q_i} \cap \Delta_X \neq \emptyset$ we consider the following two cases.

Case 1. There is a subsequence $\{k_i\colon\ i \in \omega\}$ of ω such that $\overline{Q_{k_i}} \cap \Delta_X \neq \emptyset$ for each $i \in \omega$.

In this case, since $X = \bigoplus_{\alpha\in A} X_\alpha$, for each $i \in \omega$ there is $\alpha_i \in A$ such that $\overline{Q_{k_i}} \cap \Delta_{X_{\alpha_i}} \neq \emptyset$. Thus, if we put

$$H_3 = \left\{x_\lambda y_\lambda z_\lambda\colon\ \left(y_\lambda, z_\lambda^{-1}\right) \in \bigcup_{i\in\omega} Q_{k_i}\right\},$$

then it is easy to see that $g \in \overline{H_3} \subseteq F_3(X_{\alpha(x)} \oplus \bigoplus_{i\in\omega} X_{\alpha_{k_i}}) \subseteq F_3(X)$. Since $X_{\alpha(x)} \oplus \bigoplus_{i\in\omega} X_{\alpha_{k_i}}$ is locally compact and separable, by Theorem 2.3, $F_3(X_{\alpha(x)} \oplus \bigoplus_{i\in\omega} X_{\alpha_{k_i}})$ is a k-space. Furthermore, $F_3(X_{\alpha(x)} \oplus \bigoplus_{i\in\omega} X_{\alpha_{k_i}})$ is closed in $F_3(X)$. Therefore, we have that $g \in H_3 \subseteq E$.

Case 2. There is $n \in \omega$ such that $\overline{Q_m} \cap \Delta_X = \emptyset$ for each $m \geqslant n$.

Since $g \in \overline{\{x_\lambda y_\lambda z_\lambda \in H_1\colon\ x_\lambda \in B_n\}}$, $\overline{\bigcup_{m\geqslant n} Q_m} \cap \Delta_X \neq \emptyset$. Hence there is $\alpha_0 \in A$ such that $\overline{\bigcup_{m\geqslant n} Q_m} \cap \Delta_{X_{\alpha_0}} \neq \emptyset$. It follows, from our assumption of case 2, that for every $U \in \mathcal{U}_{\alpha_0}$ $\{m \geqslant n\colon\ (Q_m \cap X_{\alpha_0}^2) \cap U \neq \emptyset\}$ is infinite, and hence

$$\overline{\bigcup\left\{P_m\colon\ \left(Q_m \cap X_{\alpha_0}^2\right) \cap U \neq \emptyset,\ m \geqslant n\right\}} \cap \Delta_{\alpha(x)} \neq \emptyset.$$

Therefore, if we put $H_4 = \{x_\lambda y_\lambda z_\lambda\colon\ (y_\lambda, z_\lambda^{-1}) \in \bigcup_{m\geqslant n}(Q_m \cap X_{\alpha_0}^2)\}$, then $g \in \overline{H_4}$. On the other hand, H_4 is contained in the k-space $F_3(X_{\alpha(x)} \oplus X_{\alpha_0})$ that is closed in $F_3(X)$. Hence we can see that $g \in H_4 \subseteq E$.

In any case, we can prove that $g \in E$. Thus E is a closed subset of $F_3(X)$. It follows that $F_3(X)$ is a k-space. □

From Proposition 3.2, 3.3 and Theorem 4.9 in [12], we obtain the following.

Theorem 3.4. *For a metrizable space X the following are equivalent*:

(1) $F_3(X)$ *is a k-space*;
(2) $A_3(X)$ *is a k-space*;
(3) i_3 *is a quotient mapping for both cases*;
(4) X *is locally compact or dX is compact.*

As a consequence of Theorem 2.3, Theorem 3.4 and the results in [12], we have the following result.

Corollary 3.5. *Let $\mathbb{R}$ be the real line, $\mathbb{Q}$ be the space of rational numbers, $J(\kappa)$ $(\kappa \geqslant \omega)$ be the hedgehog space of spine κ such that each spine is a sequence which converges to the* *center point and the topological sum C_κ of $\kappa\,(\geqslant\omega)$ many convergent sequences with their limits.*

(1) *If $X=\mathbb{R}$, then i_n is a quotient mapping for both cases, and hence both $F_n(X)$ and $A_n(X)$ are k-spaces for each $n\in\mathbb{N}$,*
(2) *If X is $\mathbb{Q}$ or $\mathbb{R}\setminus\mathbb{Q}$, then i_n is not a quotient mapping for both cases, and hence neither $F_n(X)$ nor $A_n(X)$ are k-spaces for each $n\geqslant 3$.*
(3) *If $X=J(\kappa)$ $(\kappa\geqslant\omega)$, then i_n is a quotient mapping for Abelian case, and hence $A_n(X)$ are k-spaces for each $n\in\mathbb{N}$. For non-Abelian case, however i_3 is a quotient mapping, and hence $F_3(X)$ is a k-space, i_n is not a quotient mapping, and hence $F_n(X)$ is not a k-space for some $n\geqslant 4$.*
(4) *Let $X=C_\kappa$. If $\kappa=\omega$, then i_n is a quotient mapping for both cases, and hence both $F_n(X)$ and $A_n(X)$ are k-spaces for each $n\in\mathbb{N}$. If $\kappa>\omega$, then i_3 is a quotient mapping for both cases, and both $F_3(X)$ and $A_3(X)$ are k-spaces, but i_n is not a quotient mapping for both cases, and hence neither $F_n(X)$ nor $A_n(X)$ are k-spaces for some $n\geqslant 4$.*

4. A simple description of the free group topology

In general, it is difficult to check whether a subset of $F(X)$ $(A(X))$ is open or not. In this section, we consider the following simple descriptions $(*)_F$ and $(*)_A$ of the free group topologies of $F(X)$ and $A(X)$, respectively;

$$(*)_F\ \begin{cases}\text{a set } U\subseteq F(X) \text{ is open in } F(X) \quad \text{if and only if}\\ i_n^{-1}\big(U\cap F_n(X)\big) \text{ is open in } \big(X\oplus X^{-1}\oplus\{e\}\big)^n \text{ for each } n\in\mathbb{N}.\end{cases}$$

$$(*)_A\ \begin{cases}\text{a set } U\subseteq A(X) \text{ is open in } A(X) \quad \text{if and only if}\\ i_n^{-1}\big(U\cap A_n(X)\big) \text{ is open in } \big(X\oplus -X\oplus\{0\}\big)^n \text{ for each } n\in\mathbb{N}.\end{cases}$$

Graev [4] showed that if X is compact, then both $F(X)$ and $A(X)$ have the descriptions $(*)_F$ and $(*)_A$, respectively. On the other hand, the following facts are obvious.

Fact 4.1. *Let X be a space.*

(1) *If $F(X)$ has the description $(*)_F$, then $F(X)$ has the inductive limit topology.*
(2) *If $F(X)$ has the inductive limit topology and i_n is a quotient mapping for each $n\in\mathbb{N}$, then $F(X)$ has the description $(*)_F$.*

The same are true for $A(X)$.

Now, we shall show that if $F(X)$ has the description $(*)_F$, then i_n is a quotient mapping for each $n\in\mathbb{N}$. With the same argument we can know that the same is true for $A(X)$. We begin with proving the following lemmas.

Lemma 4.2. *Let X be a space. Let $m,n\in\mathbb{N}$ with $m<n$ and $U\subseteq F_n(X)$. If $i_n^{-1}(U)$ is open in $\overline{X}^n$, then $i_m^{-1}(U\cap F_m(X))$ is open in $\overline{X}^m$. The same is true for $A(X)$.*

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_m) \in i_m^{-1}(U \cap F_m(X))$. Then $i_m(\mathbf{x}) \in U \cap F_m(X)$. Since $m < n$, put $\mathbf{y} = (y_1, y_2, \dots, y_m, y_{m+1}, \dots, y_n)$, where

$$y_i = \begin{cases} x_i & \text{if } i \leqslant m, \\ e & \text{if } m < i \leqslant n. \end{cases}$$

Then $i_n(\mathbf{y}) = i_m(\mathbf{x}) \in U$, and hence $\mathbf{y} \in i_n^{-1}(U)$. Since $i_n^{-1}(U)$ is open in $\overline{X}^n$, we can take open neighborhoods U_i of y_i in $\overline{X}$, $i = 1, 2, \dots, n$, such that $U_i = \{e\}$ for $n < i \leqslant n$ and $\mathbf{y} \in \prod_{i=1}^n U_i \subseteq i_n^{-1}(U)$. Hence $\prod_{i=1}^m U_i$ is an open neighborhood of $\mathbf{x}$ in $\overline{X}^m$ and it is easy to see that $\prod_{i=1}^m U_i \subseteq i_m^{-1}(U \cap F_m(X))$. It follows that $i_m^{-1}(U \cap F_m(X))$ is open in $\overline{X}^m$. □

Lemma 4.3. *Let X be a space. Let $m, n \in \mathbb{N}$ with $m > n$ and $U \subseteq F_n(X)$. If $i_n(U)^{-1}$ is open in $\overline{X}^n$, then $i_m^{-1}(U \cup (F_m(X) \setminus F_n(X)))$ is open in $\overline{X}^m$. The same is true for $A(X)$.*

Proof. Since $i_m^{-1}(U \cup (F_m(X) \setminus F_n(X))) = i_m^{-1}(U) \cup i_m^{-1}(F_m(X) \setminus F_n(X))$ and $i_m^{-1}(F_m(X) \setminus F_n(X))$ is open in $\overline{X}^m$, it suffices to show that for each $\mathbf{x} \in i_m^{-1}(U)$ we can find an open neighborhood of $\boldsymbol{x}$ in $\overline{X}^m$ such that it is contained in $i_m^{-1}(U \cup (F_m(X) \setminus F_n(X)))$. Without loss of generality, we may assume that $m = n + 1$.

Let $\mathbf{x} = (x_1, x_2, \dots, x_{n+1}) \in i_{n+1}^{-1}(U)$. We shall construct open neighborhoods W_i of x_i in $\overline{X}$, $i = 1, 2, \dots, n + 1$, such that $\prod_{i=1}^{n+1} W_i \subseteq i_{n+1}^{-1}(U \cup (F_{n+1}(X) \setminus F_n(X)))$. If $x_i = e$ for some $i = 1, 2, \dots, n + 1$, then we put $W_i = \{e\}$. So we may assume that each $x_i = z_i^{\varepsilon_i}$, where $z_i \in X$ and $\varepsilon_i = \pm 1$. Let U_i be an open neighborhood of z_i in X, such that

$$\begin{cases} U_i \cap U_j = \emptyset & \text{if } z_i \neq z_j, \\ U_i = U_j & \text{if } z_i = z_j. \end{cases}$$

Put

$$A = \bigl\{\mathbf{y} \in \overline{X}^{n-1}\colon\ i_{n-1}(\mathbf{y}) = i_{n+1}(\mathbf{x}),\ \operatorname{car}(\mathbf{y}) \subseteq \operatorname{car}(\mathbf{x})\bigr\}.$$

Since $i_{n+1}(\mathbf{x}) \in U \subseteq F_n(X)$ and $x_i \neq e$ for each $i = 1, 2, \dots, n + 1$, A is a nonempty finite set. Let $\mathbf{y} = (y_1, y_2, \dots, y_{n-1}) \in A$. Then $\mathbf{y} \in i_{n-1}(U \cap F_{n-1}(X))$. By Lemma 4.2, $i_{n-1}^{-1}(U \cap F_{n-1}(X))$ is open in $\overline{X}^{n-1}$. Hence, there are open neighborhoods $V(\mathbf{y}, i)$ of y_i in $X \oplus X^{-1}$, $i = 1, 2, \dots, n - 1$, such that $\prod_{i=1}^{n-1} V(\mathbf{y}, i) \subseteq i_{n-1}^{-1}(U \cap F_{n-1}(X))$. For each $i = 1, 2, \dots, n + 1$, we define an open neighborhood W_i of x_i, such as;

$$W_i = U_i^{\varepsilon_i} \cap \bigcap \bigl\{V(\mathbf{y}, j)\colon\ \mathbf{y} = (y_1, y_2, \dots, y_{n-1}) \in A,\ \ y_j = x_i\bigr\}.$$

To prove that $W = \prod_{i=1}^{n+1} W_i \subseteq i_{n+1}^{-1}(U \cup (F_{n+1}(X) \setminus F_n(X)))$, take an arbitrary $\mathbf{a} = (a_1, a_2, \dots, a_{n+1}) \in W$. We need to show the relation when $i_{n+1}(\mathbf{a}) \in F_{n-1}(X)$. Then there is $i = 1, 2, \dots, n$ so that $a_i a_{i+1} = e$. Since $a_i \in U_i^{\varepsilon_i}$ and $a_{i+1} \in U_{i+1}^{\varepsilon_{i+1}}$, $U_i \cap U_{i+1} \neq \emptyset$. So, our definition of U_i implies that $U_i = U_{i+1}$, and also $x_i x_{i+1} = e$. Put $\mathbf{y} = (y_1, y_2, \dots, y_{n-1})$, $\mathbf{b} = (b_1, b_2, \dots, b_{n-1}) \in \overline{X}^{n-1}$, where

$$y_j = \begin{cases} x_j & \text{if } j = 1, 2, \dots, i - 1, \\ x_{j+2} & \text{if } j = i, i + 1, \dots, n - 1, \end{cases}$$

$$b_j = \begin{cases} a_j & \text{if } j = 1, 2, \dots, i - 1, \\ a_{j+2} & \text{if } j = i, i + 1, \dots, n - 1. \end{cases}$$

Then it is easy to see that $\mathbf{y} \in A$ and $\mathbf{b} \in W_1 \times W_2 \times \cdots \times W_{i-1} \times W_{i+1} \times W_{i+2} \times \cdots \times W_{n+1}$. For each $j = 1, 2, \ldots, n-1$,

$$b_j = \begin{cases} a_j \in W_j \subseteq V(\mathbf{y}, j) & \text{if } j = 1, 2, \ldots, i-1, \\ a_{j+2} \in W_{j+2} \subseteq V(\mathbf{y}, j) & \text{if } j = i, i+1, \ldots, n-1. \end{cases}$$

Hence, $\mathbf{b} \in \prod_{j=1}^{n-1} V(\mathbf{y}, j) \subseteq i_{n-1}(U \cap F_{n-1}(X))$. It follows that $i_{n+1}(\mathbf{a}) = i_{n-1}(\mathbf{b}) \in U \cap F_{n-1}(X) \subseteq U$, and hence $\mathbf{a} \in i_{n+1}^{-1}(U)$. Consequently, we can see that $W \subseteq i_{n+1}^{-1}(U \cup (F_{n+1}(X) \setminus F_n(X)))$. The argument yields that $i_{n+1}(U \cup (F_{n+1}(X) \setminus F_n(X)))$ is open in $\overline{X}^{n+1}$. □

Proppposition 4.4. *Let X be a space. If $F(X)$ has the description $(*)_F$, then i_n is a quotient mapping for each $n \in \mathbb{N}$. It is true for Abelian case.*

Proof. Fix $n \in \mathbb{N}$ and let U be a subset of $F_n(X)$ such that $i_n^{-1}(U)$ is open in $\overline{X}^n$. Put $V = U \cup (F(X) \setminus F_n(X))$. Then, by Lemmas 4.2 and 4.3, we can show that $i_m^{-1}(V \cap F_m(X))$ is open in $\overline{X}^m$ for each $m \in \mathbb{N}$. Since $F(X)$ has the description $(*)_F$, we conclude that V is open in $F(X)$, in particular, U is open in $F_n(X)$. This means that i_n is a quotient mapping. □

Fact 4.1 and Proposition 4.4 yield the following result.

Theorem 4.5. *For a space X $F(X)$ has the description $(*)_F$ if and only if $F(X)$ has the inductive limit topology and i_n is a quotient mapping for each $n \in \mathbb{N}$. It is true for Abelian case.*

Since every compact subset of $F(X)$ is contained in $F_n(X)$ for some $n \in \mathbb{N}$, $F(X)$ is a k-space if and only if $F(X)$ has the inductive limit topology and $F_n(X)$ is a k-space for each $n \in \mathbb{N}$. The same is true for Abelian case. As we mentioned in Section 1, for a Dieudonné complete space X and $n \in \mathbb{N}$, i_n is a quotient mapping iff $F_n(X)$ $(A_n(X))$ is a k-space. Thus, we can obtain the following.

Theorem 4.6. *Let X be a Dieudonné complete space. Then,*

(1) *$F(X)$ has the description $(*)_F$ if and only if $F(X)$ is a k-space, and*
(2) *$A(X)$ has the description $(*)_A$ if and only if $A(X)$ is a k-space.*

Furthermore, from Theorem 2.11 and Theorem 3.7 in [2], we can obtain a characterization of a metrizable space X such that $F(X)$ $(A(X))$ has the description $(*)_F$ $((*)_A)$, respectively.

Corollary 4.7. *Let X be a metrizable space. Then,*

(1) *$F(X)$ has the description $(*)_F$ if and only if X is locally compact separable or discrete, and*

(2) *$A(X)$ has the description $(*)_A$ if and only if X is locally compact and the set of all nonisolated points of X is separable.*

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